Tutorial 7

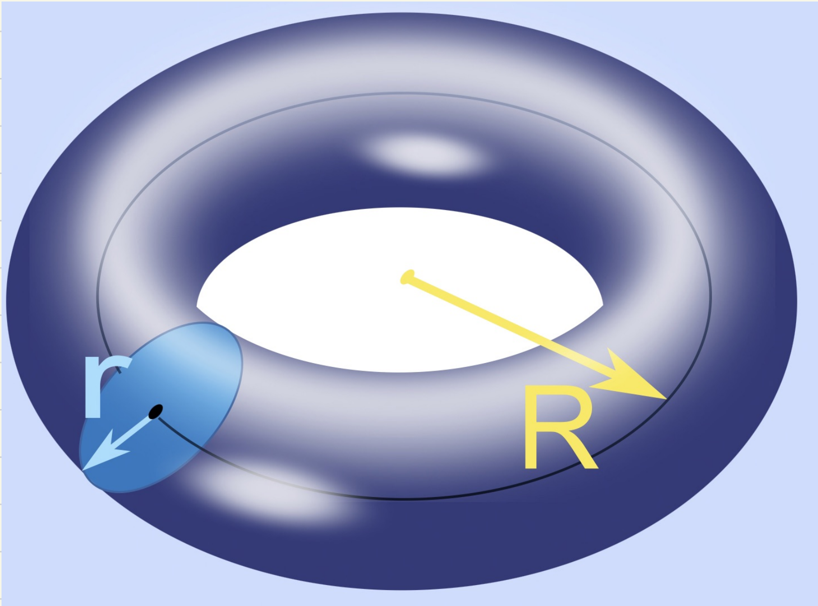
Q.1

A torus of revolution T is formed by revolving the circle $\{(x,z) : (x-R)^2 + z^2 = r^2\}$ by an angle 2π about the z-axis, where $R > r > 0$.

(a) Give a regular parametrization $\varphi : (0, 2\pi)^2 \to T$ of the torus of revolution. Check that the parametrization is indeed regular.

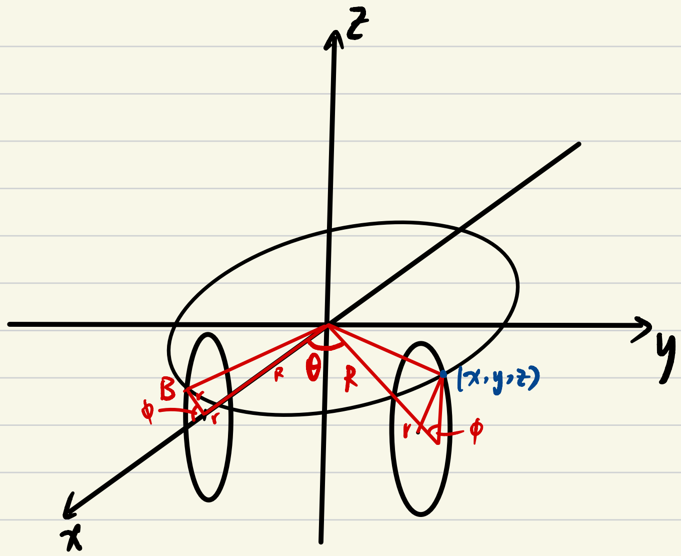
(b) Compute the outward unit normal of T at each point $(x, y, z) \in T$.

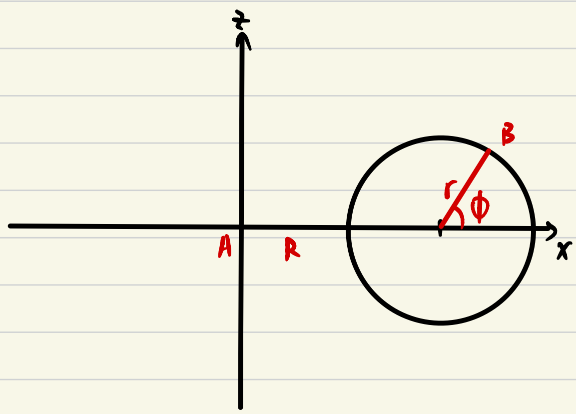
(c) Find the surface area of T.



Solution:

(a)





$B = (r\cos\phi + R, 0, r\sin\phi)$

$\therefore$ The radius of the locus of B during revolution is $r\cos\phi + R$.

$\therefore (x,y,z) = \big((r\cos\phi + R)\cos\theta, (r\cos\phi + R)\sin\theta, r\sin\phi\big)$.

$\varphi : (0,2\pi)^2 \to T$ defined by

$\varphi(\theta,\phi) = \big((r\cos\phi + R)\cos\theta, (r\cos\phi + R)\sin\theta, r\sin\phi\big)$.

is C^1 and bijective

Check regularity:

$$\varphi_\theta = \big(-(r\cos\phi+R)\sin\theta, (r\cos\phi+R)\cos\theta, 0\big)$$

$$\varphi_\phi = \big(-r\sin\phi\cos\theta, -r\sin\phi\sin\theta, r\cos\phi\big)$$

$$\varphi_\theta \times \varphi_\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -(r\cos\phi+R)\sin\theta & (r\cos\phi+R)\cos\theta & 0 \\ -r\sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi \end{vmatrix}$$

$$= r(r\cos\phi+R)\big(\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi\big)$$

$$|\varphi_\theta \times \varphi_\phi| = r\underbrace{(r\cos\phi+R)}_{\because R>r} > 0 \quad \forall (\theta,\phi)\in(0,2\pi)^2.$$

$\therefore \varphi$ is a regular parametrization of T.

(b) $$\vec{n} = \frac{\varphi_\theta \times \varphi_\phi}{|\varphi_\theta \times \varphi_\phi|} = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$$

if $(x,y,z) = \big((r\cos\phi+R)\cos\theta, (r\cos\phi+R)\sin\theta, r\sin\phi\big)$

(c) Surface area $= \int_T 1\,dS = \int_0^{2\pi}\int_0^{2\pi} r(r\cos\phi+R)\,d\theta\,d\phi$

$$= 4\pi^2 rR$$

Q.2 (Stereographic projection)

Consider the sphere $S^2 = \{(x,y,z) : x^2+y^2+z^2\}$

For any $(x,y,z) \in S^2 \setminus \{(0,0,1)\}$, let $(u,v,0)$ be the point of intersection between the plane $z=0$ & the straight line joining (x,y,z) & $(0,0,1)$ (the "north pole").

(a) Show that $(x,y,z) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right)$.

The map $\Pi : \mathbb{R}^2 \to S^2$

$$(u,v) \mapsto \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right)$$

is called the stereographic projection of the sphere S^2.

(b) Calculate the flux of the vector field

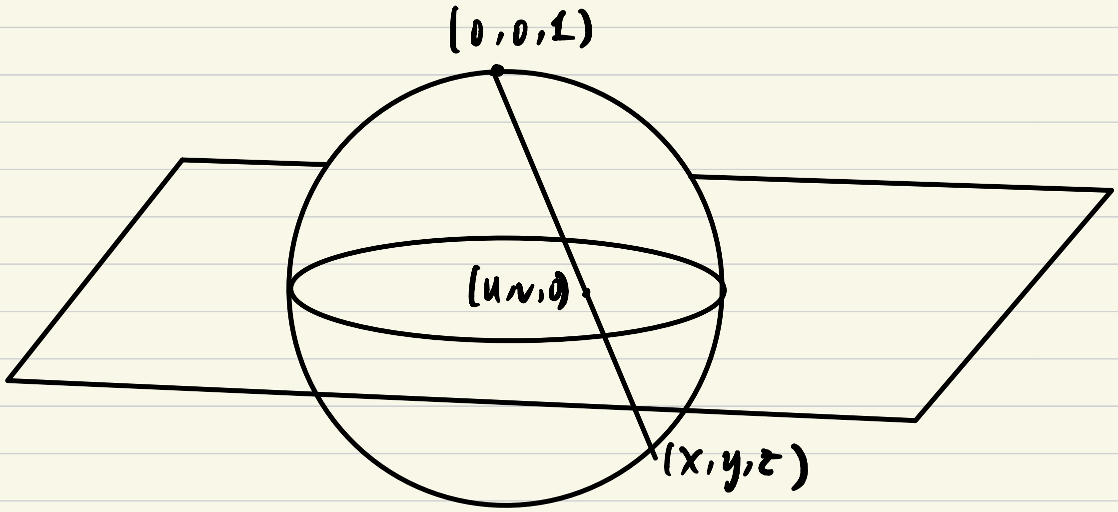
$$F(x,y,z) = (x,0,0)$$

over the unit sphere by parametrizing S^2 using

(i) the spherical coordinates;

(ii) the stereographic projection.

(a)



Equation of the straight line:

$$r(t) = (0,0,1) + t\big((u,v,0) - (0,0,1)\big)$$

$$= (tu, tv, 1-t)$$

Find the intersection:

$$(tu)^2 + (tv)^2 + (1-t)^2 = 1$$

$$(u^2+v^2+1)t^2 - 2t = 0$$

$$t = \frac{2}{u^2+v^2+1}$$

$$\therefore (x,y,z) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$$

(b) (i) The spherical coordinates:

$$\Psi(u,v) = (\sin u\cos v, \sin u\sin v, \cos u), \quad (u,v) \in (0,\pi)\times(0,2\pi).$$

$$\Psi_u = (\cos u\cos v, \cos u\sin v, -\sin u)$$

$$\Psi_v = (-\sin u\sin v, \sin u\cos v, 0)$$

$$\Psi_u \times \Psi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos u\cos v & \cos u\sin v & -\sin u \\ -\sin u\sin v & \sin u\cos v & 0 \end{vmatrix}$$

$$= \sin u(\sin u\cos v, \sin u\sin v, \cos u)$$

$$F(\Psi(u,v)) \cdot (\Psi_u \times \Psi_v) = (\sin u\cos v)(\sin^2 u\cos v) = \sin^3 u\cos^2 v$$

$$\therefore \text{Flux} = \int_0^\pi \int_0^{2\pi} \sin^3 u\cos^2 v \, dv\, du$$

$$= \left(\int_0^\pi \sin^3 u \, du\right)\left(\int_0^{2\pi} \cos^2 v \, dv\right)$$

$$= \left(-\int_0^\pi (1-\cos^2 u)\, d(\cos u)\right)\left(\frac{1}{2}\int_0^{2\pi} (1+\cos 2v)\right) dv$$

$$= \left(-\left[\cos u - \frac{\cos^3 u}{3}\right]_0^\pi\right)\left(\frac{1}{2}(2\pi)\right)$$

$$= \frac{4}{3}\pi$$

<u>Remark</u>: Technically, we are finding the flux of F across $\operatorname{Im}\Psi = S^2 \setminus (\{(x,0,0) : x \geq 0\} \cup \{(0,0,z) : z \geq 0\})$ by the def. in lecture note.

(ii) $\Pi_u = \left(\dfrac{2(-u^2+v^2+1)}{(u^2+v^2+1)^2},\ -\dfrac{4uv}{(u^2+v^2+1)^2},\ \dfrac{4u}{(u^2+v^2+1)^2} \right)$

$\Pi_v = \left(-\dfrac{4uv}{(u^2+v^2+1)^2},\ \dfrac{2(u^2-v^2+1)}{(u^2+v^2+1)^2},\ \dfrac{4v}{(u^2+v^2+1)^2} \right)$

$$\Pi_u \times \Pi_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{2(-u^2+v^2+1)}{(u^2+v^2+1)^2} & -\dfrac{4uv}{(u^2+v^2+1)^2} & \dfrac{4u}{(u^2+v^2+1)^2} \\ -\dfrac{4uv}{(u^2+v^2+1)^2} & \dfrac{2(u^2-v^2+1)}{(u^2+v^2+1)^2} & \dfrac{4v}{(u^2+v^2+1)^2} \end{vmatrix}$$

$$= \frac{1}{(u^2+v^2+1)^4}\left(-8(uv^2+u^3+u),\ -8(u^2v+v^3+v),\ 4(1-(u^2+v^2)^2)\right)$$

(Note that it is an inward pointing normal.)

$$-F(\Pi(u,v)) \cdot (\Pi_u \times \Pi_v) = \frac{16(u^2v^2+u^4+u^2)}{(u^2+v^2+1)^5}$$

$$\therefore \text{Flux} = \int_{\mathbb{R}^2} \frac{16(u^2v^2+u^4+u^2)}{(u^2+v^2+1)^5}\, dA$$

$$:= 16 \lim_{R\to\infty} \int_{B_R(0)} \frac{u^2v^2+u^4+u^2}{(u^2+v^2+1)^5}\, dA$$

$$= 16 \lim_{R\to\infty} \int_0^R \int_0^{2\pi} \frac{r^4\cos^2\theta\sin^2\theta + r^4\cos^4\theta + r^2\cos^2\theta}{(r^2+1)^5} \cdot r\, dr\, d\theta$$

$$= 16\pi \lim_{R\to\infty} \int_0^R \frac{r^5+r^3}{(r^2+1)^5}\, dr$$

$$= 16\pi \lim_{R\to\infty} \int_0^R \frac{r^3}{(r^2+1)^4}\, dr$$

$$= 16\pi \lim_{R\to\infty} \int_0^{\arctan(R)} \frac{\tan^3\phi}{\sec^8\phi} \cdot \sec^2\phi\, d\phi \qquad (r=\tan\phi)$$

$$= 16\pi \int_0^{\frac{\pi}{2}} \sin^3\phi\cos^3\phi\, d\phi$$

$$= 16\pi\left(\frac{1}{12}\right)$$

$$= \frac{4}{3}\pi$$

<u>Remark</u>: Technically, we are finding the flux of F across $\operatorname{Im}\Pi = S^2 \setminus \{(0,0,1)\}$ by the def. in lecture note.

Remark:

To get rid of the measure zero sets

$\{(x,0,0) : x \geq 0\} \cup \{(0,0,z) : z \geq 0\}$ or $\{(0,0,1)\}$ in (b)(i)(ii), we need to define what $\int_{S^2} F \cdot n \, d\sigma$ means.

Note that $\nexists$ C^1 bijective map $\varphi : U \to S^2$ since otherwise, S^2 is compact in $\mathbb{R}^3 \Rightarrow U = \varphi^{-1}(S^2)$ is compact in $\mathbb{R}^2$.

Therefore, there is no way we can define $\int_{S^2} F \cdot n \, d\sigma$ in the manner like in the lecture note.

The way out:

We can consider an open cover $\{U_1, \dots, U_n\}$ of S^2 with partition of unity $\{\psi_1, \dots, \psi_n\}$ s.t. $\exists$ parametrizations $\varphi_i : \tilde{U}_i \to U_i$ of U_i, $i = 1, \dots, n$.

Then

$$\int_{S^2} F \cdot n \, d\sigma$$

$$= \sum_{i=1}^{n} \int_{\tilde{U}_i} \left((F \circ \varphi_i) \cdot \left(\frac{\partial \varphi_i}{\partial x_i} \times \frac{\partial \varphi_i}{\partial y_i} \right) \right) \psi_i \circ \varphi_i \, dA$$

One example of such a construction is the stereographic projection $\Pi_1 : \mathbb{R}^2 \to S^2 \setminus \{(0,0,1)\}$ & the "inverted" stereographic projection $\Pi_2 : \mathbb{R}^2 \to S^2 \setminus \{(0,0,-1)\}$. (with $\psi_1 = \psi_2 \equiv \frac{1}{2}$)

(i.e. the straight line in the construction of stereographic projection start from the south pole instead)